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A LINEAR DIFFERENTIAL GAME WITH CONSTRAINTS IMPOSED ON THE CONTROL IMPULSES*

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A linear differential game with a given time of termination is considered. The terminal set of the game is determined by the condition that the phase coordinates should be equal to zero. The choice of controls is restricted by impulse constraints /1-9/. Sufficient conditions for the termination of the game which define a stable bridge are found /8, 9/. A procedure for constructing the control of the first player without using information about the amount of remaining resource of the second player is given. Classes of games for which the necessary and sufficient conditions are the same are pointed out.

1. Let us consider a game whose equations of motion have the form /10/

$$dz = N(t)du + M(t)dv,\ z \in R^n,\ u \in E_1,\ v \in E_2,\ t \in [a, p] \tag{1.1}$$

Here E_i are linear, finite-dimensional normed spaces with norms $\|x\|_i$, $x \in E_i$; $N(t)$, $M(t)$ are continuous matrices of the corresponding dimensions. The control u is chosen by the first player and the control v by the second.

The functions $w: [t, \tau] \to E_i$ with bounded variations represent the admissible program controls at every segment $[t, \tau]$. The amount of resources lost in forming controls is given by the variation $\int \|dw(r)\|_i$ /10/. Here and henceforth in Sect.1 the integration is carried out from t to τ.

The position of the game is represented by the point z, μ, ν, where the numbers $\mu \geqslant 0$, $\nu \geqslant 0$ characterize the store of the players' resources. When the controls are chosen on the segment $[t, \tau]$, the rule governing the passage of the positions is given by the formulas /10/

$$\mu(\tau) = \mu - \int \|du(r)\|_1,\ \ \nu(\tau) = \nu - \int \|dv(r)\|_2 \tag{1.2}$$

$$z(\tau) = z + \int N(r)du(r) + \int M(r)dv(r) \tag{1.3}$$

The integrals in (1.3) are regarded as the Riemann-Stieltjes integrals. The conditions that the reserve of the available resources is not exceeded can be written in the form of the following inequalities:

$$\mu(\tau) \geqslant 0,\ \ \nu(\tau) \geqslant 0 \tag{1.4}$$

The aim of the first player is to realize the equality $z(p) = 0$. The presence of control impulses leads to an instantaneous change of position, and this requires a special determination of the condition of termination /3-7/. With this purpose in mind, we shall consider the vectogram of the players

$$U(t) = \{x = N(t)u: \|u\|_1 \leqslant 1\},\ \ V(t) = \{x = M(t)v: \|v\|_2 \leqslant 1\} \tag{1.5}$$

We write the condition for the termination of the game in the form

$$z(p) + \nu(p)V(p) \subset \mu(p)U(p) \tag{1.6}$$

Let us denote the set of accessibility /10/ of the players by

$$U_t^\tau = \{x = \int N(r)du(r): \int \|du(r)\|_1 = 1\},\ U_t^t = U(t) \tag{1.7}$$

$$V_t^\tau = \{x = \int M(r)dv(r): \int \|dv(r)\|_2 = 1\},\ V_t^t = V(t) \tag{1.8}$$

The sets (1.5), (1.7) and (1.8) are convex compacta in R^n symmetrical about the origin of coordinates.

Let us denote by $\beta_1(t, \psi)$ and $\beta_2(t, \psi)$ the reference functions /11/ of the sets $U(t)$ and $V(t)$. It can be shown that the reference functions of the sets U_t^p and V_t^p are:

$$m_i(t, \psi) = \max_{t \leqslant r \leqslant p} \beta_i(r, \psi),\ i = 1, 2 \tag{1.9}$$

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The symmetry of the sets (1.5) implies that the function β_i and m_i are even in ψ.

Let us denote by (z, ψ) the scalar product of the vectors $z, \psi \in R^n$. Then the inclusion (1.6) can be written /11/ in the form

$$(z(p), \psi) + \nu(p)m_2(p, \psi) \leqslant \mu(p)m_1(p, \psi), \quad \forall\psi \in R^n \tag{1.10}$$

2. When the method of absorption of the regions of accessibility is used /1-3/, the equation of the first player is constructed for games of the form (1.1) in such a manner that the position realized at the instant t satisfies the condition

$$z + \nu V_t^p \subset \mu U_t^p \Leftrightarrow (z, \psi) + \nu m_2(t, \psi) \leqslant \mu m_1(t, \psi), \quad \forall\psi \in R^n \tag{2.1}$$

If inequality (2.1) does not hold for some vector $\psi \in R^n$, then the second player adopts the control v such, that

$$\int_t^p (M(r)\,dv(r), \psi) = \nu m_2(t, \psi), \quad \int_t^p \| dv(r) \|_2 = \nu \tag{2.2}$$

Then $\nu(p) = 0$ and we have, for any control of the first player,

$$(z(p), \psi) \geqslant (z, \psi) + \nu m_2(t, \psi) - (\mu - \mu(p))m_1(t, \psi) > \mu(p)m_1(p, \psi)$$

which means that the inequality (1.10) does not hold. We shall assume that

$$m_1(t, \psi) > 0, \quad \forall t < p, \quad \forall\psi \in R^n \tag{2.3}$$

We will give the other necessary conditions for the termination of the game. Let us fix the vector $\psi \in R^n$ and consider the one-dimensional game

$$dx = (\psi, N(t)du) + (\psi, M(t)dv), \; x = (\psi, z) \in R \tag{2.4}$$

The vectograms (1.5) for game (2.4) are represented by segments. We obtain from /7/ the necessary conditions for the termination

$$(z, \psi) + \nu m_1(t, \psi)F_1(t, \psi) \leqslant \mu m_1(t, \psi), \quad \forall\psi \in R^n \tag{2.5}$$

$$F_1(t, \psi) = \sup_{t \leqslant r \leqslant p} F_0(r, \psi); \quad F_0(t, \psi) = \frac{m_2(t, \psi)}{m_1(t, \psi)} \tag{2.6}$$

Let us write

$$F(t) = \sup_{\psi} F_1(t, \psi) \tag{2.7}$$

Lemma 1. If $\mu < \nu F(t)$, then a vector ψ can be found for any $z \in R^n$, for which inequality (2.5) will not hold.

Proof. From (2.7) it follows that $\mu < \nu F_1(t, \psi)$ for some vector $\psi \in R^n$. Therefore, using the fact that the functions (1.9) and (2.6) are even in ψ, we find that the inequality (2.5) does not hold for one of the vectors $\pm\psi$.

Let us assume that $F(t) < +\infty$ when $t < p$. From (2.6) and (2.7) it follows that

$$F(t) = \sup_{t \leqslant \tau \leqslant p} f(\tau); \quad f(\tau) = \max_{\psi} \frac{m_2(\tau, \psi)}{m_1(\tau, \varphi)} = \inf\{f \geqslant 0 : V_\tau^p \subset fU_\tau^p\} \tag{2.8}$$

Stipulating additionally the continuity of the function (2.6) and (2.7) at $t = p$, we find that $F_1(t, \psi) \leqslant F(t)$ when $t \leqslant p$.

Using the function F_1, we define the sequence of functions

$$F_{i+1}(t, \psi) = \max_{t \leqslant \tau \leqslant p} \left(F(\tau) - (F(\tau) - F_i(\tau, \psi)) \frac{m_1(\tau, \psi)}{m_1(t, \psi)} \right), \quad i \geqslant 1 \tag{2.9}$$

The above formula yields, with help of (2.7),

$$F(t) \geqslant F_{i+1}(t, \psi) \geqslant F_i(t, \psi), \; i \geqslant 1, \quad \forall\psi \in R^n \tag{2.10}$$

Theorem 1. Let

$$(z, \psi) > m_1(t, \psi)(\mu - \nu F_i(t, \psi)) \tag{2.11}$$

for some vector $\psi \in R^n$ and $i \geqslant 1$. Then the second player will be able to formulate his control so that the inequality (1.10) will not hold.

Proof. The case $i = 1$ was discussed earlier. We shall assume that the theorem holds for $i - 1$. Then from (2.11) and formula (2.9) it follows that a number $t < \tau < p$ exists for which

$$(z, \psi) > m_1(t, \psi)(\mu - \nu F(\tau)) + \nu(F(\tau) - F_{i-1}(\tau, \psi)m_1(\tau, \psi) \tag{2.12}$$

When $t \leqslant r \leqslant \tau$, the second player takes the control $dv(r) = 0$. Then, using the formulas (1.2) and (1.3) and the inequality (2.12), we can show that the following relation holds for any control of the first player:

$$(z(\tau), \psi) > m_1(t, \psi)(\mu(\tau) - \nu F(\tau)) + \nu(F(\tau) - F_{i-1}(\tau, \psi))m_1(\tau, \psi) \tag{2.13}$$

Let $\mu(\tau) \geqslant \nu F(\tau)$. Then from the inequality $m_1(t, \psi) \geqslant m_1(\tau, \psi)$ it follows that the right-hand side of inequality (2.13) is not less than $m_1(\tau, \psi)(\mu(\tau) - \nu F_{i-1}(\tau, \psi))$. According to the inductive assumption, the second player will be able to prevent the termination (1.10). If on the other hand $\mu(\tau) < \nu F(\tau)$, then the termination (1.10) will, according to Lemma 1, be impossible.

According to (2.10) we can determine

$$F_\infty(t, \psi) = \lim F_i(t, \psi), \ i \to \infty \tag{2.14}$$

The necessary conditions obtained can be written in the following form: let $S = \{\psi \in R^n : (\psi, \psi) = 1\}$. Then

$$\sup_{\psi \in S} \left(\frac{(z, \psi)}{m_1(t, \psi)} + \nu\Phi(t, \psi) \right) \leqslant \mu, \quad \Phi(t, \psi) = F_i(t, \psi), \quad i \geqslant 0 \tag{2.15}$$

Let the initial reserves of the players' resources satisfy the relation $\mu = \nu F(t)$. Then conditions (2.15) will become

$$z \in \nu K_i(t), \ K_i(t) = \{x \in R^n : (x, \psi) \leqslant (F(t) - F_i(t, \psi))m_1(t, \psi), \quad \forall \psi\} \tag{2.16}$$

3. We know /9/ that in a number of cases the conditions of regularity enable the sufficient conditions for the termination of the game to be derived from the necessary conditions.

Let us consider the conditions of regularity for the necessary conditions of the type (2.15). Let the continuous function $\Phi : S \to R$ and the convex function $m : R^n \to R$ be given, satisfying the conditions

$$m(-\psi) = m(\psi) > 0, \ \Phi(-\psi) = \Phi(\psi) \geqslant 0, \quad \forall \psi \in R^n \tag{3.1}$$

Let us write, for $x \in R^n$,

$$g(x) = \max_{\psi \in S} \left(\frac{(x, \psi)}{m(\psi)} + \Phi(\psi) \right); \quad \varphi = \max_{\psi \in S} \Phi(\psi) \tag{3.2}$$

Theorem 2. Let the maximum in the first formula of (3.2) be attained, for any vector $x \neq 0$, on a unique vector. Then

$$g(x) = g_1(x) + \varphi, \quad g_1(x) = \max_{\psi \in S} \frac{(x, \psi)}{m(\psi)} \tag{3.3}$$

Proof. Let the maximum in the second formula of (3.2) be attained on the vector ψ_1. Substituting one of the vectors $\pm\psi_1$ into the first formula of (3.2) and using the condition of evenness (3.1), we find that $g(x) \geqslant \varphi$.

We shall show that

$$g(x) = \varphi \Rightarrow x = 0 \tag{3.4}$$

Indeed, from (3.2) and (3.4) it follows that $(x, \psi_1) = 0$ and the maximum is attained on two vectors $\pm\psi_1$. Therefore $x = 0$.

The inequality $g(x) \leqslant g_1(x) + \varphi$ holds everywhere. Let $g(x) < g_1(x) + \varphi$ hold for some $x \in R^n$. We put

$$\delta = g(x) - \varphi < g_1(x); \ Y = \{y \in R^n : g_1(y) \leqslant \delta\} \tag{3.5}$$

Using the inequality $m > 0$ (3.1), we can show that the set Y is a convex compactum. Its reference function is equal to $\delta m(\psi)$.

The point $x \overline{\in} Y$. Therefore for every $y \in Y$ there exists a unique vector $\psi = \psi(y) \in S$ for which

$$g(x - y) = (x - y, \psi)/m(\psi) + \Phi(\psi) \tag{3.6}$$

The function $\psi(y)$ is continuous. Let us write

$$Z(y) = \{z \in Y : (z, \psi) = \delta m(\psi), \ \psi = \psi(y)\} \tag{3.7}$$

The set is a convex compactum depending semicontinuously on y from above. According to the Kakutani theorem /12/ there exists a fixed point $y_0 \in Z(y_0)$. Then from (3.6) and (3.7) it follows that when $\psi_0 = \psi(y_0)$, we have

$$g(x - y_0) = (x, \psi_0)/m(\psi_0) + \Phi(\psi_0) - \delta \leqslant g(x) - \delta = \varphi$$

According to (3.4), $x - y_0 = 0$, i.e. $x \in Y$ which is a contradiction.

Definition 1. We will say that the condition of regularity with the function $\Phi(t, \psi)$ holds at the instant $t < p$, if the function is continuous in ψ and the maximum in (2.5) is attained for any $z \neq 0$ $\nu > 0$ on the unique vector ψ.

Lemma 2. Let the condition of regularity hold at the instant $t < p$. Then from the necessary conditions (2.15) it follows that

$$z \in (\mu - \nu\varphi(t))U_t^p, \ \mu \geqslant \nu\varphi(t); \ \varphi(t) = \max_{\psi} \Phi(t, \psi) \tag{3.8}$$

The proof follows from inequality (2.15) and Theorem 2.

Lemma 3. Let the condition of regularity with the function $\Phi = F_0$ hold at the instant $t < p$. Then from the necessary conditions (2.15) it follows that

$$z \in (\mu - \nu F(t))U_t^p + \nu W(t), \ \mu \geqslant \nu F(t); \ W(t) = (F(t) - f(t))U_t^p \tag{3.9}$$

The proof follows from Lemmas 1 and 2 and from the form of the function F_0 (2.6), f (2.8).

Lemma 4. Let the condition of regularity with the function $\Phi = F_i$, $i \geqslant 1$ hold at the instant $t < p$. Then from the necessary conditions (2.15) it follows that

$$z \in (\mu - \nu F(t))U_t^p, \ \mu \geqslant \nu F(t) \tag{3.10}$$

The proof follows from formulas (2.7) and (2.10) and relations (3.8).

The relation $V_t^p = f(t)U_t^p$ holds for single-type games /4, 5/. In this case, from formulas (2.6) and (2.8) it follows that $F_1(t, \psi) = F(t)$. This, together with (2.15), yields conditions (3.10).

Let us consider the case when the region of accessibility of the first player at the instant t has the form

$$U_t^p = \{z: \ |(z, e_i(t))| \leqslant \alpha_i(t), \ i = 1, \ldots, n\} \tag{3.11}$$

Here the vectors $e_i(t)$ form the basis in R^n, and α_i are non-increasing functions. Let us write

$$W(t) = \{z: \ |(z, e_i(t))| \leqslant F(t) - \Phi(t, e_i(t)), \ i = 1, \ldots, n\}$$

Using the linear dependence of the vectors $e_i(t)$ and the formula (3.11), we can find that the first inclusion (3.9) follows from the necessary conditions (2.15).

4. We shall seek the sufficient conditions for termination of the game in the form

$$z \in (\mu - \nu F(t))\, U_t^p + \nu W(t), \ \ \mu \geqslant \nu F(t) \geqslant 0 \tag{4.1}$$

The necessary conditions (2.16) yield the inclusion $W(t) \subset K_i(t)$, $i \geqslant 0$. If we denote by $\stackrel{*}{-}$ the geometrical difference /13/, then from (2.16) and the form of the function (2.6) we obtain

$$W(t) \subset F(t)\, U_t^p \stackrel{*}{-} V_t^p = K_0(t) \tag{4.2}$$

Let us require that the family of sets (4.1) be a stable bridge /8, 9/. Then we can find that

$$W(t) \subset W(\tau) + (F(t) - F(\tau))\, U_t^p, \ \ t < \tau \tag{4.3}$$

Definition 2. The family of sets $W(t)$ satisfies the condition of stability, if for any $\varepsilon > 0$ and any point $r < p$ there exists a number $\delta > 0$ such that the following inclusion holds for any $t < \tau$ from the δ-neighbourhood of the point r:

$$W(t) \subset W(\tau) + (F_\varepsilon(t) - F_\varepsilon(\tau))\, U_\tau^p, \ \ F_\varepsilon(t) = F(t) + (p - t)\varepsilon \tag{4.4}$$

We note that if the matrix N and equations of motion (1.1) satisfy the local Lipshitz condition, then the condition of stability (4.4) follows from the inclusion (4.3).

It can be shown that if the family of sets $W(t)$ satisfies the inclusion (4.2) and the condition of stability (4.4), then for any $\varepsilon > 0$ and any initial instant $t_0 < p$ there exists a sequence $t_0 < t_1 < \ldots < t_i \to p$ such, that the inclusion (4.4) holds for $t = t_i$, $\tau = t_{i+1}$ and

$$W(t) + V_t^p \subset F_\varepsilon(t)\, U_\tau^p \tag{4.5}$$

Note 1. The family of sets $W(t) = 0$ satisfies the inclusion (4.2) and the condition of stability (4.4).

Definition 3. We shall call the strategy of the first player the sequence of points $t_0 < t_1 < \ldots < t_i \to p$ and the rule which places in correspondence with every triad of points z, μ, t_i, the function $u: [t_{i-1}, t_i] \to E_1$ whose variation does not exceed μ.

If the strategy of the first player and initial position $z(t_0)$, $\mu(t_0)$, $\nu(t_0)$ are given, then, when the control of the second player at the instant t_1 is chosen on the segment $[t_0, t_1]$, a position is realized determined by formulas (1.2) and (1.3). The position is determined in this manner at every instant t_i. It can be shown that this sequence of positions has a limit, and the limit is represented by the position of the instant of termination p.

Let us write for $z \in R^n$, $\mu \geqslant 0$, $t \leqslant p$, $\varepsilon \geqslant 0$

$$b(z, t, \mu, \varepsilon) = \max\{b : 0 \leqslant bF_\varepsilon(t) \leqslant \mu, z \in (\mu - bF_\varepsilon(t))U_t^p + bW(t)\} \tag{4.6}$$

If the inclusion in (4.6) does not hold for all $bF_\varepsilon(t) \leqslant \mu$, then we assume that $b(z, t, \mu, \varepsilon) = +\infty$.

Theorem 3. Let the initial state z_0, μ_0, ν_0, t_0 be such, that the following inequality holds for some $\varepsilon > 0$:

$$\nu_0 \leqslant b(z_0, t_0, \mu_0, \varepsilon) = b_0 < +\infty \tag{4.7}$$

Then a strategy of the first player will exist guaranteeing the termination (1.6).

Proof. Using the number $\varepsilon > 0$, we construct a sequence of numbers $t_0 < t_1 < \ldots < t_i \to p$, such that inclusions (4.4) and (4.5) hold for $t = t_i$, $\tau = t_{i+1}$.

Let us describe the rule used in constructing the control $u: [t_i, t_{i+1}] \to E_1$. If $b = b(z, t_i, \mu, \varepsilon) = +\infty$, then we take any admissible control. Let $b < +\infty$. Then from (4.6) it follows that

$$z = x + y, \quad x \in (\mu - bF_\varepsilon(t_i))\, U_{t_i}^p, \quad y \in bW(t_i) \tag{4.8}$$

Consider the problem of the momenta /10/

$$\varphi = \int_{t_i}^{p} \| du(r) \|_1 \to \min, \quad x + \int_{t_i}^{p} N(r)\, du(r) = 0 \tag{4.9}$$

Let φ_0 and u_0 be the solution of problem (4.9). Then

$$\begin{gathered}\varphi_0 \leqslant \mu - bF_\varepsilon(t_i) \\ x + \int_{t_i}^{t_{i+1}} N(r)\, du_0(r) \in (\mu - bF_\varepsilon(t_i) - \int_{t_i}^{t_{i+1}} \| du_0(r) \|_1)\, U_{t_{i+1}}^p\end{gathered} \tag{4.10}$$

The first inequality in (4.10) follows from (4.8). We take $u_0 : [t_i, t_{i+1}] \to E_1$ as the control of the first player.

Let us assume that the position z_i, μ_i, ν_i realized at the instant t_i satisfies the inequality

$$\nu_i \leqslant b(z_i, t_i, \mu_i, \varepsilon) = b_i < +\infty \tag{4.11}$$

Condition (4.8) will hold for $z = z_i$, $\mu = \mu_i$, $b = b_i$. Suppose that the second player has chosen the control $v : [t_i, t_{i+1}] \to E_2$, and has lost q of his resources. Then using the inclusions (4.4), (4.5) and (4.8), we obtain

$$\begin{gathered}y + \int_{t_i}^{t_{i+1}} M(r)\, dv(r) \in b_i W(t_i) + qV_{t_i}^p \subset q(W(t_i) + V_{t_i}^p) + \\ (b_i - q)\, W(t_i) \subset qF_\varepsilon(t_i) U_{t_{i+1}}^p + (b_i - q)\, W(t_i)\end{gathered}$$

and this will yield, with help of (4.10) and of the first relation of (4.8),

$$\begin{gathered}z_{i+1} \in (\mu_{i+1} - (b_i - q)\, F_\varepsilon(t_{i+1}))\, U_{t_{i+1}}^p + (b_i - q)\, W(t_{i+1}) \\ \mu_{i+1} \geqslant (b_i - q)\, F_\varepsilon(t_{i+1})\end{gathered}$$

Therefore the inequality (4.11) will hold for the position realized at the instant t_{i+1} and $b_{i+1} \geqslant b_i - q \geqslant \nu_i - q = \nu_{i+1}$.

From the inequality (4.11) and inclusion (4.5) we can find that $z_i + \nu_i V_{t_i}^p \subset \mu_i U_{t_i}^p$, and this implies that the inclusion (1.6) will hold for the limiting position.

Passing to the limit, we can find from Theorem 3 that conditions (4.1) define a stable bridge /8, 9/. We can construct a strategy of the first player /8/ extremal with respect to the bridge, but this algorithm utilises the quantity ν. The algorithm for constructing the control u according to the rule (4.9), does not utilize this information.

5. We can use the following procedure /14/ to construct the sets $W(t)$:

$$W_0(t) = K_i(t), \quad W_{k+1}(t) = \bigcap_{t \leqslant \tau \leqslant p} (W_k(\tau) + (F(t) - F(\tau))\, U_t^p) \tag{5.1}$$

$$W(t) = \bigcap_{k\geq 1} W_k(t)$$

Here the number $i \geqslant 0$ is fixed. We note that $W_{k+1}(t) \subset W_k(t)$. From this we can find /14/ that the inclusion (4.3) holds.

Let us consider a game with region of accessibility (3.11). We shall seek the sets $W(t)$ in the form (3.11), with the functions $\alpha_i(t)$ replaced by $y_i(t)$. Then the set appearing on the right-hand side of the inclusion (4.4) will have the form

$$\{z \in R^n: \ |(z,\ e_i(\tau))| \leqslant y_i(\tau) + (F(t) - F(\tau) + (\tau - t)\,\varepsilon)\,\alpha_i(\tau),\ \ i = 1, \ldots, n\} \tag{5.2}$$

From the linear independence of the vectors e_i it follows that $e_i(\tau) = a_{i1}(\tau, t)e_1(t) + \ldots + a_{in}(\tau, t)\,e_n(t)$, and this, taking (5.2) into account, implies that the inclusion (4.4) will hold if

$$\sum_{j=1}^{n} |a_{ij}(\tau, t)|\, y_j(t) \leqslant y_i(\tau) + (F(t) - F(\tau) + (\tau - t)\,\varepsilon)\,\alpha_i(\tau) \tag{5.3}$$

It can be shown that the inclusion (4.2) will hold if

$$y_i(t) \leqslant F(t)\,\alpha_i(t) - m_2(t,\ e_i(t)),\quad i = 1, \ldots, n \tag{5.4}$$

For the class of games under discussion, the function f (2.8) has the form

$$f(\tau) = \max_{1\leqslant i\leqslant n} \frac{m_2(\tau, e_i(\tau))}{\alpha_i(\tau)} \tag{5.5}$$

Example /3/. Let $z \in R^2$ and let

$$e_1(t) = (0,\ 1),\quad e_2(t) = (2,\ t - p),\quad \alpha_1(t) = 1,\quad \alpha_2(t) = p - t$$

in formula (3.11). The vectogram of the second player has the form

$$V(t) = \{(x_1,\ x_2) \in R^2: x_1 = v\sin(p - t),\ \ x_2 = v\cos(p - t),\ |v| \leqslant 1\}$$

Using (5.5) we can show that

$$F(t) = \frac{m_2(t, e_2(t))}{\alpha_2(t)} = 2\,\frac{\sin(p - t)}{p - t} - \cos(p - t),\quad p - \alpha \leqslant t \leqslant p$$

$$F(t) = F(p - \alpha),\quad t \leqslant p - \alpha;\quad 2\operatorname{ctg}\alpha = 2\alpha^{-1} - \alpha,\quad 0 < \alpha < \pi$$

From this it follows that $F(t) > 1 = m_2(t, e_1, (t))$ when $t < p$. The inequalities (5.3) take the form

$$y_1(t) \leqslant y_1(\tau) + F(t) - F(\tau) + (\tau - t)\,\varepsilon \tag{5.6}$$

$$(\tau - t)\,y_1(t) + y_2(t) \leqslant y_2(\tau) + (F(t) - F(\tau))(p - \tau) + (\tau - t)\,\varepsilon\,(p - \tau)$$

It can be shown that the functions $y_1(t) = \min(F(t) - 1;\ (t - p)\,F^{\cdot}(t))$, $y_2(t) = 0$, in the neighbourhood of every point $r \leqslant p$, satisfy inequalities (5.6), and satisfy conditions (5.4) for every $t \leqslant p$.

REFERENCES

1. KRASOVSKII N.N., On the problem of pursuit. PMM, 27, 2, 1963.
2. KRASOVSKII N.N., REPIN YU.M. and TRET'YAKOV V.E., On some game-theoretic situations in the theory of control systems. Izv. Akad. Nauk SSSR, Tekhn. Kibernetika, 4, 1965.
3. KRASOVSKII N.N. and TRET'YAKOV V.E., On the pursuit problem in the case of constraints on the control momenta. Differents. Uravneniya, 2, 5, 1966.
4. POZHARITSKII G.K., Impulsive tracking in the case of second-order linear objects of the same type, PMM 30, 5, 1966.
5. KRASOVSKII N.N., Game-theoretical encounter of motions with bounded impulses. PMM 32, 2, 1968.
6. POZHARITSKII G.K., On the problem of impulsive encounter of motions. PMM, 35, 5, 1971.
7. UKHOBOTOV V.I., On a differential game with impulse controls. Sb. in-ta mekh. mekh.-mat. fak-ta MGU, 1, 1973.
8. SUBBOTINA N.N. and SUBBOTIN A.I., On an alternative for a differential game of encounter-evasion with constraints on the momenta of the player's controls. PMM, 39, 3, 1975.
9. KRASOVSKII N.N. and SUBBOTIN A.I., Positional Differential Games. Moscow, Nauka, 1974.
10. KRASOVSKII N.N., Theory of Motion Control, Moscow, Nauka, 1968.
11. PSHENICHNYI B.N., Convex Analysis and Extremal Problems. Moscow, Nauka, 1980.

12. BONENBLAST KH.F. and KARLIN S., On a theorem of Ville. In: Infinite Antagonistic Games. Moscow, Fixmatgiz, 1963.
13. PONTRYAGIN L.S., On the linear differential games. 2. Dokl. Akad. Nauk USSR, 175, 4, 1967.
14. UKHOBOTOV V.I., On the construction of a stable bridge in a retention game. PMM, 45, 2, 1981.

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ASYMPTOTIC TRAJECTORIES AND THE STABILITY OF THE PERIODIC MOTIONS OF AN AUTONOMOUS HAMILTONIAN SYSTEM WITH TWO DEGREES OF FREEDOM*

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The existence of motions asymptotic to the periodic trajectories of a Hamiltonian system with two degrees of freedom is studied. It is assumed that the Hamiltonian function is time-independent and analytic in the neighbourhood of the periodic trajectories. It is noted that, under certain constraints, the conditions for the existence of asymptotic trajectories are equivalent to the conditions for orbital instability of the limiting periodic motion. As an application, the asymptotic trajectories in the problem of the motion of a dynamically symmetric rigid body relative to the centre of mass in a central Newtonian gravitational field in a circular orbit and in the problem of the motion of a heavy rigid body with a fixed point are considered.

1. Isoenergetic reduction. Let a generalized conservative system with two degrees of freedom have a T-periodic motion, distinct from the equilibrium position, and in the neighbourhood of the closed trajectory of the phase space corresponding to this periodic motion (PM), let the Hamiltonian function H be analytic.

Two characteristic exponents of the system of equations of the perturbed motion, linearized in the neighbourhood of the periodic motion, are always (in the case of an autonomous Hamiltonian system) equal to zero. If the other two characteristic exponents have a non-zero real part, then the PM is orbitally unstable. If they are pure imaginary (equal to $\pm i\alpha$), then both orbital instability and stability are possible, depending on the type of non-linear terms in the equations of the perturbed motion. In fact, if $k\alpha \neq n\omega$ ($\omega = 2\pi/T$; $k = 1, 2, 3, 4$; n is an integer), we usually have orbital stability; the case $k = 1, 2$ correspond to the boundary of the domains of orbital stability to a first approximation, while with $k = 3, 4$ orbital instability is possible inside these domains. A similar description of the conditions for stability and instability may be found in /1, 2/; we shall merely remark here that they are the same as the stability and instability conditions at the isoenergetic level $H = h =$ **const**, at which the trajectory of the PM considered lies.

To solve the problem of the existence of trajectories asymptotic to the PM trajectory, we observe that the asymptotic trajectories must correspond to the same value of the constant h as does the PM trajectory. At this fixed energy level, the equations of motion (Whittaker equations) have the form of the Hamilton equations /3/. Let us obtain these.

We can always choose /4/ (though in general this is extremely difficult) the canonical conjugate variables q_i, p_i $(i = 1, 2)$ in such a way that the PM corresponds to their values

$$q_1 = \omega t + q_{10}, \quad p_1 = q_2 = p_2 = 0 \qquad (1.1)$$

where t is time, and q_{10} is the initial value of the coordinate q_1. The Hamiltonian function is then 2π-periodic in q_1.

It can be assumed without loss of generality that the trajectory of the PM (1.1) lies at the zero energy level $H = 0$. The Hamiltonian function can be expanded in a converging series

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